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Quantization of the Chern-Simons coupling constant

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Abstract: We investigate the quantum consistency of p-form Maxwell-Chern-Simons electrodynamics in $3p+2$ spacetime dimensions (for p odd). These are the dimensions where the Chern-Simons term is cubic, i.e., of the form $F \wedge F \wedge A$. For the theory to be consistent at the quantum level in the presence of magnetic and electric sources, we find that the Chern-Simons coupling constant must be quantized. We compare our results with the bosonic sector of eleven dimensional supergravity and find that the Chern-Simons coupling constant in that case takes its corresponding minimal allowed possibility.

Keywords: Chern-Simons Theories, M-Theory, p-branes.

Contents

1. Introduction

Yang-Mills gauge theory in $2+1$ dimensions with a Chern-Simons (CS) term (also called topological mass term) is known to have its topological mass quantized for gauge groups with a non-trivial third homotopy group [1]. The argument goes as follows. The CS term added to the Yang-Mills action preserves local gauge invariances for spacetime manifolds without boundary, but is not invariant under large gauge transformations. Under large gauge transformations, the total action varies by a term proportional to the instanton number representing the class of $\pi_3(G)$. If we ask for the invariance of the path integral under such topologically non-trivial gauge transformations, we get a quantization condition of the CS coupling constant.

In general, the CS coupling constant is quantized since $\pi_3(G) \simeq \mathbb{Z}$ for any compact connected simple Lie group G [2]. However, in the abelian case with compact group manifold $G = \mathrm{U}(1)$ all the homotopy groups higher than π_1 are trivial and, a priori, the CS coupling constant can take arbitrary values. The authors of [3] have pointed out that when electric and magnetic charges are present the CS coupling constant must be quantized just as in the non-abelian case. The quantization arises from two key features. The first, is that in the presence of magnetic sources the electric charge is non-conserved for non-vanishing CS coupling constant, hence electric worldlines may end on magnetic sources. The value of the electric charge created or annihilated on a magnetic charge is related to the CS coupling constant. The second key property is the usual Dirac charge quantization condition that was shown to remain valid in the presence of a CS term. It was also pointed out in [3] that their result may be straightforwardly generalized to p-form electrodynamics in $(2p+1)$-dimensional spacetime, where the electric $(p-1)$-branes have the same dimensionality than the Dirac brane attached to the magnetic $(p-2)$-brane.

The quantization of the abelian CS coupling constant was rederived in [4] for spacetimes with topology $S^1 \times M^2$ in the presence of a non-vanishing total magnetic flux on M^2. Since the spacetime manifold contains a one-cycle S^1, a quantization condition is expected from the fact that $\pi_1(\mathrm{U}(1)) \simeq \mathbb{Z}$. Similar phenomena occur at finite temperature [5]. In this case we are interested in euclidean spacetimes, where the time direction is effectively compactified into a circle.

All the analysis cited above considered odd-dimensional spacetimes with linear equations of motion. The next step would be to consider abelian theories with a cubic CS term in the action. This is the subject of the present paper. The non-abelian case has been considered in CS gravity, and it also leads to the quantization of the CS coupling constant, which corresponds, in that case, to the quantization of Newton's constant [6]. For p-form electrodynamics, a cubic term exists only in $(3p+2)$-dimensional spacetime. Furthermore, if we want a non-vanishing CS term p has to be odd since the gauge group is abelian. The extended objects carrying electric charge in these theories are $(p-1)$-branes. Magnetic charge is carried by spacelike $2p-1$-dimensional extended objects.

Cubic CS terms in p-form electrodynamics appear, for instance, in the bosonic sector of many supergravity theories. An example of such a theory is the celebrated eleven-dimensional supergravity [7], which is currently believed to describe the low energy effective action of M-theory. Five-dimensional supergravity also contains a CS term. This theory is known to resemble $D = 11$ supergravity in many respects [8] and could be used as a toy model to test various ideas of M-theory in a simpler setting (see also [9] for another five-dimensional toy model). This similarity arises from the fact that $D = 5$ supergravity can be realized as a specific truncation of a Calabi-Yau compactification of $D = 11$ supergravity [10].

Because of its simplicity, the paper will consider only five-dimensional Maxwell-Chern-Simons (MCS) electrodynamics, which is the first example of the theories we are looking for. This case is analogous to the bosonic sector of $D = 5$ simple supergravity in a fixed gravitational background. The analysis will use differential forms so that generalization to higher dimensions is straightforward.

2. A 5D model of a gauge field with a Chern-Simons term

MCS theory coupled to electric sources may be described, in five-dimensional spacetime $\mathcal{M}_5$ by the following action,

$$I = \frac{1}{2}\int F \wedge {}^*F + \frac{\alpha}{6}\int F \wedge F \wedge A - \int A \wedge {}^*J_e\,, \tag{2.1}$$

where A is the gauge field, $F = dA$ its curvature, J_e the electric current and α a dimensionfull constant called the CS coupling constant. The electric sources are pointlike objects, which, by analogy with eleven-dimensional supergravity, we may call them M0-branes. The equations of motion and Bianchi identity are

$$d{}^*F + \frac{\alpha}{2}F \wedge F = {}^*J_e \tag{2.2}$$

$$dF = 0\,. \tag{2.3}$$

The action is gauge invariant if and only if J_e is a conserved current, i.e., if $d{}^*J_e = 0$. This requirement is consistent with the equations of motion, as can be seen by taking the exterior derivative of (2.2), which gives precisely (2.3). We may take the electric current to be generated by the 1-dimensional worldvolume, $\mathcal{M}_1$, of a particle of charge e, parameterized by $z^\mu(\tau)$. The dual of its associated current may be written as ${}^*J_e = e\,P(\mathcal{M}_1)$, where the four-form $P(\mathcal{M}_1)$ is the Poincaré dual of $\mathcal{M}_1$. (In appendix A we introduce the definition and some key properties of Poincaré duality. For more details on this subject and its use in electrodynamics see [11].)

Now we introduce magnetic sources. In 5D, ${}^*F$ is a 3-form, and therefore these will correspond to one-dimensional extended objects: "magnetic strings", or M1-branes. The magnetic string current is described by a 2-form, J_m. The field equations now take the form

$$d{}^*F + \frac{\alpha}{2}F \wedge F = {}^*J_e \tag{2.4}$$

$$dF = {}^*J_m\,. \tag{2.5}$$

These equations are gauge invariant, but the electric current J_e is not, in general, conserved. In fact, taking the exterior derivative of (2.4) gives

$$d{}^*J_e = \alpha{}^*J_m \wedge F\,. \tag{2.6}$$

This last equation is telling us that electric current does not need to be conserved on the worldsheet of a magnetic string, $\mathcal{M}_2$, unless the flux of the field strength across $\mathcal{M}_2$ is required to vanish. Since the magnetic current J_m is conserved, its worldsheet, $\mathcal{M}_2$, is a two-dimensional surface with no boundaries. The dual of the magnetic charge may be written: ${}^*J_m = g\,P(\mathcal{M}_2)$, where g is a constant measuring the charge of the magnetic source.

Note that the present situation is different from both the standard Maxwell case and the quadratic Chern-Simons studied in [3]. Indeed, eq. (2.6) shows that we cannot specify external sources, because the conservation rule for J_e depends explicitly on the field strength F. This is a major obstacle in the construction of a variational principle giving rise to eqs. (2.4) and (2.5). One solution would be to consider the manifold $\mathcal{M}_2$ to be a surface where boundary conditions must be prescribed. However, here we adopt a different strategy, inspired by eleven-dimensional supergravity. We will add degrees of freedom living on the magnetic string worldsheet. These new degrees of freedom will dynamically introduce the "boundary conditions" for F. In the next section we will present an action principle giving rise to equations (2.4) and (2.5).

3. The action

Consider the modified Bianchi identity (2.5). Following the standard procedure, we introduce a Dirac worldvolume, $\mathcal{N}_3$, whose boundary is the magnetic string worldvolume, $\mathcal{M}_2 = \partial\mathcal{N}_3$, and solve it by writing the field strength as $F = dA +^* G$. Here $^*G = P(\mathcal{N}_3)$, and therefore $^*J_m = d^*G$ (see appendix A). On the magnetic string worldvolume, $\mathcal{M}_2$, we shall also introduce a scalar field Φ, which describes the new degrees of freedom discussed in the previous section. Consider now the following action principle in 5 spacetime dimensions:

$$\begin{aligned} I = \frac{1}{2}\int {}^*F\wedge F + \frac{\alpha}{6}\int dA\wedge dA\wedge A + \frac{\alpha}{2}\int {}^*G\wedge A\wedge dA - \\ - \int A\wedge {}^*J_e + \frac{\alpha}{2}\int\left(-f\wedge A + \frac{1}{2}{}^*f\wedge f + \omega\Phi^*j\right)\wedge {}^*J_m + I_k\,. \end{aligned} \tag{3.1}$$

Here

$$f = d\Phi - A\,|_{\text{pullback}} +^* g_{-1}\,, \tag{3.2}$$

where $A\,|_{\text{pullback}}$ denotes the pullback of A on $\mathcal{M}_2$, and * is the Hodge star on the worldvolume $\mathcal{M}_2$. From now on, we will omit the subscript "pullback" which should be self-evident from the context. The parameter ω is any real number and I_k are kinetic terms. Finally, g_{-1} defines a set of Dirac worldlines that originate instantons (localized at one spacetime event) on the magnetic string worldsheet. The case of one instanton of strength ν located at the point $\mathcal{M}_0$ (denoted in that way to make higher dimensional generalization straightforward) is described by the dyonic instanton "current" $^*j = \nu\, P(\mathcal{M}_0)$, a 0-form proportional to the Poincaré dual to $\mathcal{M}_0$ in $\mathcal{M}_2$. The Dirac point is defined by a worldline $\mathcal{N}_1$ ending or originating at the instanton, in such a way that $^*j = d^*g$ if $^*g = \nu\, P(\mathcal{N}_1)$.

Note that, for simplicity of notation, sometimes we make use of $d\Phi$ and $^*g_{-1}$ in the action as 5-dimensional forms. These should be understood as arbitrary extensions whose pullback gives the corresponding 2-dimensional differential form on the string worldsheet.

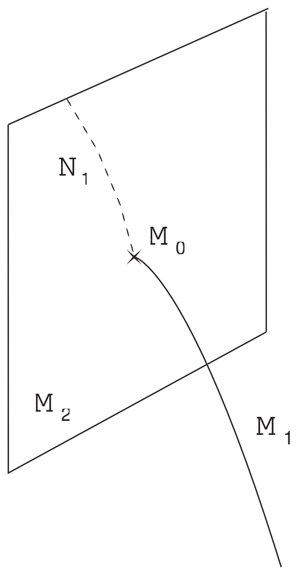

Figure 1: An electric worldline, $\mathcal{M}_1$, ends on a magnetic worldsheet $\mathcal{M}_2$ and produces a magnetic instanton, $\mathcal{M}_0$.

JHEP01(2003)054

4. Gauge invariances

The action (3.1) is invariant under two different gauge transformations. First,

$$A \longrightarrow A + d\Lambda \tag{4.1}$$

$$\Phi \longrightarrow \Phi + \Lambda, \tag{4.2}$$

where, in the second equation, Λ is understood as the pullback along the string. In the absence of magnetic sources, this transformation reduces to the standard gauge transformation of electrodynamics. In the present case, the invariance of the action under (4.1) and (4.2) requires that the following identity is satisfied:

$$d^*J_e = \frac{\alpha}{2}(1-\omega)\, {}^*j \wedge {}^*J_m\,. \tag{4.3}$$

This expression shows that electric charge does not need to be conserved if $\omega \neq 1$. In fact, an electric worldline may end on a magnetic worldsheet and create a dyonic instanton on it. The eleven-dimensional analog is the well known fact that M2-branes can end on M5-brane with a dyonic string as intersection. In that case, a spacelike picture is possible because there is “more room” in eleven dimensions. Also note that in the absence of instantons on the worldsheet of the string, we get conservation of the electric charge for any value of ω. The Poincaré dual translation of the relation (4.3) is $\partial\mathcal{M}_1 = \mathcal{M}_0 \subset \mathcal{M}_2$ since ${}^*j \wedge {}^*J_m$ is proportional to the Poincaré dual of $\mathcal{N}_1$ in $\mathcal{M}_5$ (${}^*J_m$ first projects on $\mathcal{M}_2$ and then ${}^*j$ projects on $\mathcal{N}_1$ in $\mathcal{M}_2$).

The second gauge freedom of this system is associated with the position of the Dirac brane and can be described as follows. We may deform $\mathcal{N}_3$ into a new manifold $\mathcal{N}_3'$ sharing the same boundary $\mathcal{M}_2$. This new manifold is equally appropriate for solving (2.5) in the way indicated above. The field strength F is invariant under this displacement of the Dirac brane, but

$${}^*G' = {}^*G + d{}^*V\,, \tag{4.4}$$

$$A' = A - {}^*V\,. \tag{4.5}$$

Here, ${}^*V = gP(\mathcal{V})$ is the one-form dual to the manifold $\mathcal{V}$ swept while moving the Dirac worldvolume from $\mathcal{N}_3$ to $\mathcal{N}'_3$, that is, $\partial\mathcal{V} = \mathcal{N}_3' - \mathcal{N}_3$. The field f on the magnetic string is also invariant under (4.5) because the pullback of ${}^*V$ vanishes. The action (3.1) is invariant under small displacements of the Dirac brane, that is, displacements such that $\mathcal{V}$ does not intersect the worldvolume of any other object. This shows that the Dirac membrane is unobservable.

The same remark applies to the Dirac point living on the magnetic string. The corresponding gauge transformation is

$${}^*g \to {}^*g + d{}^*v\,,\ \Phi \to \Phi - {}^*v\,. \tag{4.6}$$

The scalar ${}^*v$ is proportional to the Poincaré dual of the surface described by the Dirac worldline in the string worldsheet. The action is obviously invariant under (4.6).

JHEP01(2003)054

5. Equations of motion

The equation of motion coming from varying A is

$$d^*F + \frac{\alpha}{2} F \wedge F - \frac{\alpha}{2}{}^*J_m \wedge ({}^*f + f) = {}^*J_e \, . \tag{5.1}$$

Note that on deriving this equation we have made use of the following identity, which is discussed in appendix B,

$${}^*G \wedge {}^*G = 0 \, . \tag{5.2}$$

If conveniently regularized, this identity holds at any point where the Dirac worldvolumes do not intersect, which will be the case in general, because, as we shall see below, consistency of the variational principle will require that Dirac membranes never intersect themselves (Dirac veto).

Varying Φ we obtain,

$$(dA + d^*f + \omega^*j) \wedge {}^*J_m = 0 \, . \tag{5.3}$$

As a consistency check, note that taking the exterior derivative of (5.1) and using (5.3) we get (4.3). From the definition of f, we also have, on $\mathcal{M}_2$, the Bianchi identity

$$df = -dA + {}^*j \, . \tag{5.4}$$

Since the magnetic worldsheet is a minkowskian two-dimensional manifold, the 1-form field strength f defined on it can be decomposed into its self-dual part f_+ and anti-self-dual part f_-,

$$f = f_+ + f_- \, , \qquad {}^*f_\pm = \pm f_\pm \, . \tag{5.5}$$

Equation (5.1) can then be rewritten,

$$d^*F + \frac{\alpha}{2} F \wedge F - \alpha^*J_m \wedge f_+ = {}^*J_e \, , \tag{5.6}$$

while expressions (5.3) and (5.4) are equivalent to

$$df_+ + dA = \frac{1-\omega}{2}\, {}^*j \, , \tag{5.7}$$

$$df_- = \frac{1+\omega}{2}\, {}^*j \, . \tag{5.8}$$

These last two equations are defined on the magnetic worldsheet. We notice that the anti-self-dual part is decoupled from the bulk field A, since f_- does not appear in (5.6) nor in (5.7).

When $\omega = -1$, it is therefore natural to consider the sector of the theory where $f_- = 0$, in which case the dyonic instanton is self-dual. Furthermore, the equations (5.6) and (5.7) then describe the bosonic sector of simple $D = 5$ supergravity [8]. The worldsheet self-dual boson is precisely the analog of the dynamics of the M5-brane in eleven-dimensional supergravity. Strictly speaking, for these theories, the self-duality condition must arise from the variational principle. Indeed, it is possible to write an action similar to (3.1)

JHEP01(2003)054

for which $f_- = 0$ arises as a consequence of the variation of the action, using the PST technique [12]. A regularized action has been proposed recently in [13]. Concerning the issue of regularization in MCS theories with cubic CS term, one may look also at [9].

Note however that one may consider other sectors of the theory for arbitrary values of ω. For instance, the sector $f_+ = 0$ leads to the equation (2.4) which does not contain explicitly the magnetic membrane fields. Still, the presence of the magnetic source imposes boundary conditions on F via equation (5.7). In particular, when $\omega = +1$, on the brane, the electric field is orthogonal to the magnetic string.

6. Charge conservation

It is a well known fact that the "brane source charge" is not conserved in theories with CS terms (for an extensive discussion on this, see [14]). In the present case, as we have already noticed, the electric source charge current, J_e is not conserved. Nevertheless, we know that the action (3.1) is gauge invariant in the standard way. We therefore expect a conserved charge associated to this symmetry. We may discover this conservation law from the equations of motion. In fact, from (4.3) we see that

$$d\left({}^*J_e - \frac{\alpha}{2}(1-\omega)\,{}^*g \wedge {}^*J_m\right) = 0\,. \tag{6.1}$$

This can be rephrased by saying that the Dirac worldline is the geometric continuation of the electric worldline in the magnetic string worldsheet. The electric charge is therefore "transferred" into the Dirac point. This is exactly the same phenomena that was described previously in three spacetime dimensions [3].

Now we can define the following conserved charge,

$$Q = \int_{D^4} ({}^*J_e - \frac{\alpha}{2}(1-\omega)\,{}^*g \wedge {}^*J_m) \tag{6.2}$$

Here D^4 is a four-dimensional, spacelike ball, whose boundary $\partial D^4 = S^3$. is a three-sphere. Note that although Q is a conserved quantity in virtue of (6.1), it is not gauge invariant. In fact, we can always deform the Dirac worldvolume so that, for instance, if initially it crossed D^4, at the end it will no longer cross it. To define a gauge invariant charge, we must define Q in the limit where the radius of S^3 goes to infinity.

Let us now take two different representatives, $D^4_{(1)}$ and $D^4_{(2)}$. The first one does not intersect the magnetic worldsheet while the second one has no intersection with the worldline. If we evaluate the tension with $D^4_{(1)}$ we get

$$\int_{D^4_{(1)}} {}^*J_e = e$$

and with $D^4_{(2)}$

$$\begin{aligned}\frac{\alpha}{2}(1-\omega)\int_{D^4_{(2)}} {}^*g \wedge {}^*J_m &= \frac{\alpha g}{2}(1-\omega)\int_{S^1} {}^*g = \frac{\alpha g}{2}(1-\omega)\int_{B^2} {}^*j \\ &= \frac{\alpha g \nu}{2}(1-\omega)\end{aligned}$$

where $D^4_{(2)} \cap \mathcal{M}_2 = S^1 = \partial B^2$. In conclusion, the electric charge e, created (or annihilated) on a instanton of charge ν, located on a magnetic string of charge g is given by

$$e = \frac{\alpha g \nu}{2}(1-\omega)\,. \tag{6.3}$$

Note now that the Bianchi identity (5.7) implies that

$$f_+ = d\Phi_+ - A + \frac{1-\omega}{2}\,{}^*g\,. \tag{6.4}$$

This last equation combined with (5.6) leads to

$$d\left({}^*F + \frac{\alpha}{2}A\wedge dA + \alpha A\wedge {}^*G + \alpha\Phi_+{}^*J_m\right) = {}^*J_e + \frac{\alpha}{2}(1-\omega){}^*J_m\wedge {}^*g\,. \tag{6.5}$$

from where we can rewrite Q in terms of quantities defined at spatial infinity only,

$$Q = \int_{S^3_\infty} {}^*F + \frac{\alpha}{2}A\wedge dA - g\alpha\int_\gamma A - \alpha g\left(\Phi(+\infty) - \Phi(-\infty)\right)\,. \tag{6.6}$$

Here γ is the line where the Dirac membrane intersects the sphere S^3_∞. The quantities $\Phi(\pm\infty)$ are the values of the scalar field Φ at the points where the magnetic string intersects S^3_∞, where the sign in $\pm\infty$ is defined by the orientation of the string.

7. Dirac vetos

Let us now vary the action with respect to the Dirac worldvolume. If we first fix its boundary (the string worldsheet) we obtain,

$$d^*F + \frac{\alpha}{2}dA\wedge dA\Big|_{\text{pullback on Dirac worldvolume}} = 0\,. \tag{7.1}$$

For consistency with (5.6) this implies three Dirac vetos

- *No electric charge can be located on the Dirac worldvolume.* This is the standard Dirac veto.
- *No magnetic charge can be located on the Dirac worldvolume.*
- *Dirac worldvolumes cannot intersect each other (or themselves).* In other words, the pullback of ${}^*G$ on the Dirac worldvolume must vanish, so that dA may be replaced by F in (7.1). Note that the Dirac vetos, in the presence of the CS term, are more restrictive than in the Maxwell case where only the first one applies.

From the variation of the magnetic string we obtain the Lorentz force equation acting on the magnetic strings. This equation will depend on the precise form of the kinetic terms I_k present in the action (3.1).

If we vary the action with respect to the Dirac wordline with the instanton position fixed, we get

$$d({}^*f + A)|_{\text{pullback on Dirac worldine}} = 0\,, \tag{7.2}$$

which implies, for consistency with (5.3), a Dirac veto on the string worldsheet:

- *No dyonic charge can be located on the Dirac worldine.* This is only necessary if $\omega \neq 0$.

JHEP01(2003)054

8. Quantization conditions

8.1 Quantization condition in the bulk

Consider a finite displacement of a given Dirac membrane, described, as in section 4, by the manifold $\mathcal{V}$ swept by the Dirac worldvolume as it is moved from $\mathcal{N}_3$ to $\mathcal{N}_3'$, so that $\partial\mathcal{V} = \mathcal{N}_3' - \mathcal{N}_3$. We again define the Poincaré dual of this manifold, ${}^*\!V = P(\mathcal{V})$. The variation of the action is given by

$$\delta_V I = \int {}^*\!V \wedge {}^*J_e \, . \tag{8.1}$$

Note that the terms coming from integrating over Dirac membranes, Dirac points, magnetic strings or instantons vanish. This is due to the Dirac vetos. In fact, $\mathcal{V}$ is a 4-dimensional, compact, manifold. If there were, say, a magnetic string worldsheet, $\mathcal{M}_2$, inside it (a non-compact 2-dimensional manifold), then an intersection between $\mathcal{M}_2$ and $\partial\mathcal{V}$ is unavoidable. But $\partial\mathcal{V}$ describes the initial and final configurations of the magnetic string, and therefore the intersection implies that either the final or the initial configuration (or both) cannot satisfy the Dirac veto. The same phenomena occurs for Dirac membranes. Now we can easily integrate (8.1) to get,

$$\delta_V I = egk \, , \tag{8.2}$$

where $k \in \mathbb{Z}$ is the number of electric worldlines crossing through $\mathcal{V}$. Obviously, for an infinitesimal variation we may always keep $\mathcal{V}$ free of electric intersections, hence this anomaly appears only for finite gauge transformations.

If we require the path integral to be invariant under (8.2), we obtain the usual Dirac quantization condition

$$eg = 2\pi n \tag{8.3}$$

for any integer n (we have set $\hbar = 1$).

8.2 Quantization condition on the worldsheet

In the case of the instanton the computation of the Dirac anomaly is subtle, among other things because the instanton is dyonic. The Dirac anomaly is,

$$\delta_v I = \frac{\alpha g \omega}{2} \int_{\mathcal{M}_2} {}^*\!v \, {}^*\!j \, . \tag{8.4}$$

where ${}^*\!v = \nu P(\mathcal{V}_2)$, and $\mathcal{V}_2$ is defined, on the magnetic string worldsheet, by the variation of the Dirac point worldline. Taking into account the fact that the instanton is dyonic (and therefore lives on $\mathcal{V}_2$), it can be shown that consistency at the quantum level imposes

$$\alpha g \nu^2 \omega = 2\pi m \tag{8.5}$$

with m an integer. The quantization of an instanton in two dimensions was previously considered in [15]. As one can see, there is a subtle factor of two with respect to the "naive" application of Dirac quantization condition with equal electric and magnetic charge. This

JHEP01(2003)054

feature is generic for dyons [16] (in dimensions 2 mod 4), and comes from the fact that the phase e^{-iI} must be strictly invariant only under gauge transformations connected to the identity. The simplest system for which this consideration applies contains two dyons, which multiplies by two the total flux (for details see [16]).

8.3 Quantization of the CS coupling constant

From (6.3), (8.3) and (8.5) we obtain

$$\alpha = \frac{\omega}{(1-\omega)^2} \frac{e_0^3}{\pi^2} N \,, \qquad (N \in \mathbb{Z}) \,, \tag{8.6}$$

which is the announced quantization condition of the CS coupling constant α, where e_0 is the minimal electric charge of the MCS theory. The integer N is directly related to the (quantized) electric charge created by the minimal instanton charge ($m = 1$) living on the string with minimal magnetic charge $2\pi/e_0$. This formula is only valid when $\omega \neq 0, 1$. When $\omega = 0$ the right hand side of (8.4) identically vanishes, and therefore the quantization condition in the worldsheet, (8.5), is not required. The parameter α can therefore take any value in this case. When $\omega = 1$, the electric charge is conserved, and therefore we cannot relate, through (6.3), the value of electric and magnetic sources. Again, in this case, α is arbitrary.

When $\omega = -1$ and $f_- = 0$, the system is (on-shell) the 5-dimensional analog of the MCS sector of 11-dimensional supergravity coupled to M2- and M5-branes. The coupling constant α, in that case, is fixed by the requirement of supersymmetry, and it turns out to be given by the choice $N = 1$ in (8.6).

9. Conclusions

In the present paper we first constructed a gauge invariant action principle which generalizes the usual MCS theory (for cubic CS terms) so that it can be coupled to magnetic sources. It turns out that there is no way to implement that by attaching Dirac branes to the magnetic sources only. It is necessary — as it is the case when eleven-dimensional supergravity is coupled to M5-branes — to add further degrees of freedom living on the magnetic source. Then we studied the quantum consistency of the different possible theories (parameterized by a real parameter, ω), and conclude that the CS coupling constant must be quantized according to (8.6). Although we have used the 5-dimensional case throughout this paper, the generalization for higher (5 modulo 6) dimensional spacetime is straightforward. One only needs to substitute the different form fields by their analogs of higher degree.

Bachas [17] obtained previously a similar quantization condition[1]

$$\alpha = -\frac{e_0^3}{(2\pi)^2} M \,, \qquad (M \in \mathbb{Z}) \,. \tag{9.1}$$

[1] Compare the relation (9.1) with equation (4.22) of [17]. The different normalizations can be translated into $\alpha = \sqrt{2}k\kappa_{(5)}$ and $e_0 = \sqrt{2}\kappa_{(5)}q$.

His derivation was based on a different argumentation, using compactification to four dimensions together with the Witten effect. The two quantization conditions (8.6) and (9.1) coincide only for $\omega = -1$.

Condition (9.1) was obtained even earlier under the assumption that F was a non-trivial cocycle with integer periods [18], that is, in the absence of magnetic sources. Indeed, since the five-dimensional spacetime manifold $\mathcal{M}_5$ is closed, it can be taken as the boundary of a six-dimensional manifold $\mathcal{M}_6$. Therefore the CS term $F \wedge F \wedge A$ is lifted to $F \wedge F \wedge F$. Furthermore, physics should not depend on the choice of the auxiliary manifold $\mathcal{M}_6$. Under the assumption that the change $\delta\mathcal{M}_6$ of six-dimensional manifold is the direct product of three two-cycles on which F takes integer periods, the condition (9.1) follows from the invariance of the path integral.

However, as pointed out by Witten [19], a CS term is quantum-mechanically inconsistent when M is not a multiple of 6 because the integral of $F \wedge F \wedge F$ is not a multiple of six for arbitrary closed manifolds $\delta\mathcal{M}_6$. In other words, the insertion of the action (2.1) in the path integral may lead to inconsistencies if the U(1) bundle of the MCS theory is non-trivial. Quantum consistency is restored by adding the gravitational and fermionic sector of eleven-dimensional supergravity, as shown in [19], by taking into account gravitational corrections plus a rather subtle argument using E_8 gauge theory (see also [20] for subsequent developments).

We stress that, in contrast, we derived here (8.6) from the presence of magnetic sources (thus F is *not* closed) but with a *trivial* U(1) bundle and a topologically *trivial* spacetime manifold (i.e. $\mathcal{M}_5 \simeq \mathbb{R}^5$), thereby avoiding the above-mentioned quantum inconsistency. Surprisingly enough, we still obtain the supergravity factor in this extremely simple situation.

Note added in proof: while finishing the present article, the work [21] appeared, which consider similar issues regarding the construction of the action principle. This is done from a different perspective, and in the framework of M-theory.

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JHEP01(2003)054

A. Review of Poincaré duality

Let $\mathcal{M}_p$ be a smooth, oriented manifold of dimension p embedded on a D-dimensional manifold $\mathcal{M}$. Let x^μ, $\mu = 1, \ldots, D$ be coordinates on $\mathcal{M}$. We parameterize $\mathcal{M}_p$ with p coordinates σ^i by $x^\mu = z^\mu(\sigma^i)$ and define the following p-tensor on $\mathcal{M}$

$$V^{\mu_1 \cdots \mu_p}(x) = \int_{\mathcal{M}_p} \delta^{(D)}(x - z) dz^{\mu_1} \cdots dz^{\mu_p} \,. \tag{A.1}$$

The Poincaré dual of $\mathcal{M}_p$, $P(\mathcal{M}_p)$, is the $(D - p)$-form defined by

$$P(\mathcal{M}_p) = {}^*V \,. \tag{A.2}$$

The key, defining property of the Poincaré dual is that given any p-form on $\mathcal{M}$, Ω, then

$$\int_{\mathcal{M}_p} \Omega \mid_{\text{pullback}} = - \int_{\mathcal{M}} P(\mathcal{M}_p) \wedge \Omega \tag{A.3}$$

where $\Omega \mid_{\text{pullback}}$ is the pullback of Ω on $\mathcal{M}_p$. Another important property of the Poincaré dual is the following: If $\partial \mathcal{M}_p$ is the boundary of $\mathcal{M}_p$, then

$$P(\partial \mathcal{M}_p) = (-1)^{D-p+1} dP(\mathcal{M}_p) \,. \tag{A.4}$$

Both (A.3) and (A.4) can be derived straightforwardly from (A.1), (A.2).

B. Regularization of ${}^*G \wedge {}^*G$

In this appendix we show that the singular expression

$${}^*G(x) \wedge {}^*G(x) \,, \tag{B.1}$$

may be set locally to zero by properly regularizing the delta functions present on it. The idea is to give some small transverse width ϵ to the Dirac worldvolume, and then take the limit $\epsilon \to 0$. For this purpose, we will consider, in a neighborhood of the point x, a local set of coordinates on spacetime such that the Dirac worldvolume is parallel to the coordinates (x^0, x^1, x^2). The dual of expression (B.1) is a 1-form whose components are proportional to

$$\epsilon_{\mu\nu\rho\sigma\tau} \int \delta^{(5)}(x - y) dy^\alpha \wedge dy^\mu \wedge dy^\nu \int \delta^{(5)}(x - \tilde{y}) d\tilde{y}^\rho \wedge d\tilde{y}^\sigma \wedge d\tilde{y}^\tau \,. \tag{B.2}$$

Here both integrals are taken over the worldvolume of the Dirac membrane, which may be parameterized by σ^a $(a = 0, 1, 2)$, setting $y^\mu \equiv y^\mu(\sigma^a)$. In the local set of coordinates we chose, we can write $y^a = \sigma^a$, $y^3 = y^4 = 0$. The transversal width of the delta function is implemented by taking

$$\delta^{(5)}(x - y(\sigma)) \longrightarrow \delta(x^a - \sigma^a) \Delta_\epsilon(x^3) \Delta_\epsilon(x^4) \,, \tag{B.3}$$

where Δ_ϵ is any regular function which is equal to the delta function as ϵ vanishes. The deltas in the longitudinal directions are kept unchanged, and can be integrated so that expression (B.2) is proportional to

$$\epsilon_{abcde} \left(\Delta(x^3) \Delta(x^4) \right)^2 \tilde{\epsilon}^{\alpha ab} \tilde{\epsilon}^{cde} \,, \tag{B.4}$$

JHEP01(2003)054

where $\tilde{\epsilon}^{abc}$ is the Levi-Civita symbol of the Dirac 3-dimensional worldvolume. It is clear that this quantity is zero for any finite value of ϵ, and in particular in the limit $\epsilon \to 0$. Let us mention that the previous argument can also be applied to any product of Poincaré duals taken at the same point. Indeed, any such kinds of identities (e.g. ${}^{*}V(x) \wedge {}^{*}V(x)$) have been implicitly set to zero everywhere in this paper.

Let us now mention some topological subtleties. First of all, the expression (A.1) is, strictly speaking, only well-defined locally [22]. Secondly, the above proof of (5.2) was essentially local, a point directly linked to the previous one. A way out of this problem is to consider the following "framing" regularization procedure (see [23, pp. 284–285], and references therein): Let $P(\mathcal{M}_p) \wedge P(\mathcal{M}_p)$ be a product of Poincaré duals taken at the same point. We replace it by the non-singular product $P(\mathcal{M}_p) \wedge P(\mathcal{M}_p^\epsilon)$, where $\mathcal{M}_p^\epsilon$ is a manifold (i) "close" to $\mathcal{M}_p$ the maximal separation of which is measured by ϵ, and (ii) without any intersection with $\mathcal{M}_p$: $\mathcal{M}_p \bigcap \mathcal{M}_p^\epsilon = \emptyset$. Eventually, one takes the limit of vanishing ϵ.

This framing procedure should also be applied to the action itself for every appearing singular product. For instance the cubic CS term $dA \wedge dA \wedge A$ also needs to be regularized since A contains a delta-like singularity. Therefore, three non-intersecting branes are required to regularize the CS action: the original Dirac brane worldsheet, $\mathcal{M}_2$, and two (auxiliary) displaced worldsheets, $\mathcal{M}_2^\epsilon$ and $\mathcal{M}_2^{\epsilon'}$ ($\epsilon \to 0$, $\epsilon' \to 0$).

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JHEP01(2003)054

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JHEP01(2003)054